Equitable Colorings of Planar Graphs without Short Cycles

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Abstract

An *equitable coloring* of a graph is a proper vertex coloring such that the sizes of every two color classes differ by at most 1. Chen, Lih, and Wu conjectured that every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd cycle or Δ is odd and $G = K_{\Delta,\Delta}$. Nakprasit proved the conjecture holds for planar graphs with maximum degree at least 9. Zhu and Bu proved that the conjecture holds for every C_3-free planar graph with maximum degree at least 8 and for every planar graph without C_4 and C_5 with maximum degree at least 7.

In this paper, we prove that the conjecture holds for planar graphs in various settings, especially for every C_3-free planar graph with maximum degree at least 6 and for every planar graph without C_4 with maximum degree at least 7, which

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improve or generalize results on equitable coloring by Zhu and Bu. Moreover, we prove that the conjecture holds for every planar graph of girth at least 6 with maximum degree at least 5.

Key Words: Equitable coloring; Planar graph; Cycle; Girth

1 Introduction

Throughout this paper, all graphs are finite, undirected, and simple. We use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$, and $\delta(G)$, respectively, to denote vertex set, order, edge set, size, maximum degree, and minimum degree of a graph G. We write $xy \in E(G)$ if x and y are adjacent. The graph obtained by deleting an edge xy from G is denoted by $G - \{xy\}$. For any vertex v in $V(G)$, let $N_G(v)$ be the set of all neighbors of v in G. The *degree of* v, denoted by $d_G(v)$, is equal to $|N_G(v)|$. We use $d(v)$ instead of $d_G(v)$ if no confusion arises. For disjoint subsets U and W of $V(G)$, the number of edges with one end in U and another in W is denoted by $e(U, W)$. We use $G[U]$ to denote the subgraph of G induced by U.

An *equitable k-coloring* of a graph is a proper vertex k-coloring such that the sizes of every two color classes differ by at most 1. We say that G is *equitably k-colorable* if G has an equitable k-coloring.

It is known [2] that determining if a planar graph with maximum degree 4 is 3-colorable is NP-complete. For a given n-vertex planar graph G with maximum degree 4, let G' be a graph obtained from G by adding $2n$ isolated vertices. Then G is 3-colorable if and only if G' is equitably 3-colorable. Thus, finding the minimum number of colors need to color a graph equitably even for a planar graph is an NP-complete problem.

Hajnal and Szemerédi [4] settled a conjecture of Erdős by proving that every graph G with maximum degree at most Δ has an equitable k-coloring for every $k \geq 1 + \Delta$. In its 'complementary' form this result concerns decompositions of a sufficiently dense graph into cliques of equal size. This result is now known as Hajnal

and Szemerédi Theorem. Later, Kierstead and Kostochka [6] gave a simpler proof of Hajnal and Szemerédi Theorem in the direct form of equitable coloring. The bound of the Hajnal-Szemerédi theorem is sharp, but it can be improved for some important classes of graphs. In fact, Chen, Lih, and Wu [1] put forth the following conjecture.

Conjecture 1. *Every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd cycle or Δ is odd and $G = K_{\Delta,\Delta}$.*

Lih and Wu [8] proved the conjecture for bipartite graphs. Meyer [9] proved that every forest with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$ colors. This result implies conjecture holds for forests. The bound of Meyer is attained at the complete bipartite $K_{1,m}$: in every proper coloring of $K_{1,m}$, the center vertex forms a color class, and hence the remaining vertices need at least $m/2$ colors. Yap and Zhang [13] proved that the conjecture holds for outerplanar graphs. Later Kostochka [5] extended the result for outerplanar graphs by proving that every outerplanar graph with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$. Again this bound is sharp.

In [14], Zhang and Yap essentially proved the conjecture holds for planar graphs with maximum degree at least 13. Later Nakprasit [10] extended the result to all planar graphs with maximum degree at least 9.

Other studies focused on planar graphs without some restricted cycles. Li and Bu [7] proved that the conjecture holds for every planar graph without C_4 and C_6 with maximum degree at least 6. Zhu and Bu [15] proved that the conjecture holds for every C_3-free planar graph with maximum degree at least 8 and for every planar graph without C_4 and C_5 with maximum degree at least 7. Tan [11] proved that the conjecture holds for every planar graph without C_4 with maximum degree at least 7. Unfortunately the proof contains some flaws.

In this paper, we prove that each graph G in various settings has an equitably m-colorable such that $m \leq \Delta$. Especially we prove that the conjecture holds for planar graphs in various settings, especially for every C_3-free planar graph with maximum

degree at least 6 and for every planar graph without C_4 with maximum degree at least 7, which improve or generalize results on equitable coloring by Zhu and Bu [15]. Moreover, we prove that the conjecture holds for every planar graph of girth at least 6 with maximum degree at least 5.

2 Preliminaries

Many proofs in this paper involve edge-minimal planar graph that is not equitably m-colorable. The minimality is on inclusion, that is, any spanning subgraph with fewer edges is equitably m-colorable. In this section, we describe some properties of such graph that appear recurrently in later arguments. The following fact about planar graphs in general is well-known and can be found in standard texts about graph theory such as [12].

Lemma 1. *Every planar graph G of order n and girth g has $e(G) \leq (g/(g-2))(n-2)$. Especially, a C_3-free planar graph G has $e(G) \leq 2n-4$ and $\delta(G) \leq 3$.*

Let G be an edge-minimal C_3-free planar graph that is not equitably m-colorable with $|G| = mt$, where t is an integer. As G is planar and without C_3, a graph G has an edge xy where $d(x) = \delta \leq 3$. By edge-minimality of G, the graph $G - \{xy\}$ has an equitable m-coloring ϕ having color classes $V_1', V_2', \ldots, V_m'$. It suffices to consider only the case that $x, y \in V_1'$. Choose $x \in V_1'$ such that x has degree δ and order $V_1', V_2', \ldots, V_\delta'$ in a way that $N(x) \subseteq V_1' \cup V_2' \cup \cdots \cup V_\delta'$. Define $V_1 = V_1' - \{x\}$ and $V_i = V_i'$ for $1 \leq i \leq m$.

We define $\mathcal{R}$ recursively. Let $V_1 \in \mathcal{R}$ and $V_j \in \mathcal{R}$ if there exists a vertex in V_j which has no neighbors in V_i for some $V_i \in \mathcal{R}$. Let $r = |\mathcal{R}|$. Let A and B denote $\bigcup_{V_i \in \mathcal{R}} V_i$ and $V(G) - A$, respectively. Furthermore, we let A' denote $A \cup \{x\}$ and B' denote $B - \{x\}$. From definitions of $\mathcal{R}$ and $B, e(V_i, \{u\}) \geq 1$ for each $V_i \in \mathcal{R}$ and $u \in B$. Consequently $e(A, B) \geq r[(m-r)t + 1]$ and $e(A', B') \geq r(m-r)t$.

Suppose that there is k such that $k \geq \delta + 1$ and $V_k \in \mathcal{R}$. By definition of $\mathcal{R}$, there exist $u_1 \in V_{i_1}, u_2 \in V_{i_2}, \ldots, u_s \in V_{i_s}, u_{i_{s+1}} \in V_{i_{s+1}} = V_k$ such that $e(V_1, \{u_1\}) =$

$e(V_{i_1}, \{u_2\}) = \cdots = e(V_{i_s}, \{u_{s+1}\}) = 0$. Letting $W_1 = V_1 \cup \{u_1\}, W_{i_1} = (V_{i_1} \cup \{u_2\}) - \{u_1\}, \ldots, W_{i_s} = (V_{i_s} \cup \{u_{s+1}\}) - \{u_s\}$, and $W_k = (V_k \cup \{x\}) - \{u_{s+1}\}$, otherwise $W_i = V_i$, we get an equitable m-coloring of G. This contradicts to the fact that G is a counterexample.

Thus, in case of C_3-free planar graph, we assume $\mathcal{R} \subseteq \{V_1, V_2, \ldots, V_\delta\}$ where $\delta \leq 3$ is the minimum degree of non-isolated vertices.

We summarize our observations here.

Observation 2. *If G is an edge-minimal C_3-free planar graph that is not equitably m-colorable with order mt, where t is an integer, then we may assume*
(i) $\mathcal{R} \subseteq \{V_1, V_2, \ldots, V_\delta\}$ where $\delta \leq 3$ is the minimum degree of non-isolated vertices;
(ii) $e(u, V_i) \geq 1$ for each $u \in B$ and $V_i \in \mathcal{R}$;
(iii) $e(A, B) \geq r[(m-r)t + 1]$ and $e(A', B') \geq r(m-r)t$.

3 Results on C_3-free Planar Graphs

First, we introduce some useful tools and notation that will be used later.

Theorem 3. *[3] (Grötzsch, 1959) If G is a C_3-free planar graph, then G is 3-colorable.*

Lemma 4. *Let m be a fixed integer with $m \geq 1$. Suppose that any C_3-free planar graph of order mt with maximum degree at most Δ is equitably m-colorable for any integer $t \geq k$. Then any C_3-free planar graph with order at least kt and maximum degree at most Δ is also equitably m-colorable.*

Proof. Suppose that any C_3-free planar graph of order mt with maximum degree at most Δ is equitably m-colorable for any integer $t \geq k$. Consider a C_3-free planar graph G of order $mt + r$ where $1 \leq r \leq m - 1$ and $t \geq k$. If $r = m - 1$ or $m - 2$, then $G \cup K_{m-r}$ is equitably m-colorable by hypothesis. Thus also is G. Consider $r \leq m - 3$. Let x be a vertex with minimum degree d. We assume that $G - \{x\}$ is equitably m-colorable to use induction on r. Thus the coloring of $G - \{x\}$ has $r + 1$

color classes with size $t-1$. Since there are at most d forbidden colors for x where $d \leq 3$, we can add x to a color class of size $t-1$ to form an equitable m-coloring of G. This completes the proof $\square$

Lemma 5. *[1] If G is a graph with maximum degree $\Delta \geq |G|/2$, then G is equitably Δ-colorable.*

Observation 6. *By Lemmas 4 and 5, for proving that the conjecture holds for C_3-free planar graphs it suffices to prove only C_3-free planar graphs of order Δt where $t \geq 3$ is a positive integer.*

Lemma 7. *[14] Let G be a graph of order mt with chromatic number χ such that $\chi \leq m$, where t is an integer. If $e(G) \leq (m-1)t$, then G is equitably m-colorable.*

Lemma 8. *Suppose G is a C_3-free planar graph with $\Delta(G) = \Delta$. If G has an independent s-set V' and there exists $U \subseteq V(G) - V'$ such that $|U| > s(1+\Delta)/2$ and $e(u, V') \geq 1$ for all $u \in U$, then U contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$.*

Proof. Let U_1 consist of vertices in U with exactly one neighbor in V'. If $r = |U_1|$, then $r + 2(|U| - r) \leq \Delta s$ which implies $r \geq 2|U| - \Delta s > s$. Consequently, V' contains a vertex γ which has at least two neighbors in U_1. Since G is C_3-free, this two neighbors are not adjacent. Thus U_1 contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$. $\square$

Lemma 9. *[10] If a graph G has an independent s-set V' and there exists $U \subseteq V(G) - V'$ such that $e(u, V') \geq 1$ for all $u \in U$, and $e(G[U]) + e(V', U) < 2|U| - s$, then U contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$.*

Notation. *Let $q_{m,\Delta,t}$ denote the maximum number not exceeding $2mt - 4$ such that each C_3-free planar graph of order mt, where t is an integer, is equitably m-colorable if it has maximum degree at most Δ and size at most $q_{m,\Delta,t}$.*

The next Lemma is similar to that in [10] except that we use V_1 instead of V_1' which is erratum. Nevertheless later arguments in [10] stand correct.

Lemma 10. *Let G be an edge-minimal C_3-free planar graph that is not equitably m-colorable with order mt, where t is an integer, and maximum degree at most Δ. If $e(G) \leq (r+1)(m-r)t - t + 2 + q_{r,\Delta,t}$, then B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1$.*

Proof. If $e(G[A']) \leq q_{r,\Delta,t}$, then $G[A']$ is equitably r-colorable. Consequently, G is equitably m-colorable. So we suppose $e(G[A']) \geq q_{r,\Delta,t} + 1$. By Observation 2, $e(A'-V_1', B') \geq (r-1)(m-r)t$. Note that $e(G[B]) = e(G[B'])$, $e(V_1, B) = e(V_1', B')+1$. So $e(G[B])+e(V_1, B) = e(G[B'])+e(V_1', B')+1 = e(G)-e(G[A'])-e(A'-V_1', B')+1 < 2mt - 2rt - t + 3 = 2|B| - |V_1|$. By Lemma 9, B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1$. □

Lemma 11. *Let G be an edge-minimal C_3-free planar graph that is not equitably m-colorable with order mt, where t is an integer, and maximum degree at most Δ. If B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1$, then $e(G) \geq r(m-r)t + q_{r,\Delta,t} + q_{m-r,\Delta,t} - \Delta + 4$.*

Proof. Suppose $e(G) \leq r(m-r)t + q_{r,\Delta,t} + q_{m-r,\Delta,t} - \Delta + 3$. If $e(G[A']) \leq q_{r,\Delta,t}$, then $G[A']$ is equitably r-colorable. Consequently, G is equitably m-colorable. So we suppose $e(G[A']) \geq q_{r,\Delta,t} + 1$. This with Observation 2 implies $e(G[A']) + e(A, B') \geq q_{r,\Delta,t} + 1 + r(m-r)t$. Note that $e(G[A']) + e(A, B') = e(G[A]) + e(A, B)$. Let $A_1 = (A - \{\gamma\}) \cup \{\alpha, \beta\}$ and $B_1 = (B \cup \{\gamma\}) - \{\alpha, \beta\}$. Then $e(G[A_1]) + e(A_1, B_1) \geq e(G[A]) + e(A, B) - \Delta + 2 \geq q_{r,\Delta,t} + 1 + r(m-r)t - \Delta + 2$. So $e(G[B_1]) = e(G) - e(G[A_1]) + e(A_1, B_1) \leq q_{m-r,\Delta,t}$ which implies $G[B_1]$ is equitably $(m-r)$-colorable. Combining with $(V_1 - \{\gamma\}) \cup \{\alpha, \beta\}, V_2, \ldots, V_r$, we have G equitably m-colorable which is a contradiction. □

Corollary 12. *Let G be an edge-minimal C_3-free planar graph that is not equitably m-colorable with order mt, where t is an integer, and maximum degree at most Δ.*

Then $e(G) \geq r(m-r)t + q_{r,\Delta,t} + q_{m-r,\Delta,t} - \Delta + 4$ *if one of the following conditions are satisfied:*
(i) $(m-r)t + 1 > (t-1)(1+\Delta)/2$;
(ii) $e(G) \leq (r+1)(m-r)t - t + 2 + q_{r,\Delta,t}$.

Proof. This is a direct consequence of Lemmas 8, 10, and 11. $\square$

Now we are ready to work on C_3-free planar graphs.

Lemma 13. *(i)* $q_{1,\Delta,t} = 0$. *(ii)* $q_{2,\Delta,t} \geq 3$ *for* $t \geq 3$. *(iii)* $q_{3,\Delta,t} \geq 2t$.

Proof. (i) and (ii) are obvious. (iii) is the result of Theorem 3 and Lemma 7. $\square$

Lemma 14. $q_{4,\Delta,t} \geq \min\{q_{3,\Delta,t} + 3t + 3 - \Delta, 4t - \Delta + 9\}$ *for* $\Delta \geq 5$ *and* $t \geq 3$.

Proof. Condider $\Delta \geq 5$ and $t \geq 3$. Suppose G' is a C_3-free planar graph with maximum degree at most Δ and $e(G') \leq \min\{q_{3,\Delta,t} + 3t + 3 - \Delta, 4t - \Delta + 9\}$ but G' is not equitably 4-colorable. Let $G \subseteq G'$ be an edge-minimal graph that is not equitably 4-colorable. From Table 1, $e(G) > e(G')$. This contradiction completes the proof.

r	lower bounds on size	Reasons
3	$q_{3,\Delta,t} + 3t + 3 - \Delta$ or $q_{3,\Delta,t} + 3t + 2$	Corollary 12(ii), Lemma 13
2	$4t - \Delta + 9$ or $5t + 5$	Corollary 12(ii), Lemma 13
1	$q_{3,\Delta,t} + 3t + 3 - \Delta$ or $5t + 2$	Corollary 12(ii), Lemma 13

Table 1: Lower bounds on size of G in the proof of Lemma 14

$\square$

Lemma 15. $q_{5,\Delta,t} \geq \min\{q_{3,\Delta,t} + 6t + 6 - \Delta, q_{4,\Delta,t} + 4t + 3 - \Delta, 7t + 2\}$ *for* $\Delta \geq 5$ *and* $t \geq 3$.

Proof. Use Table 2 for an argument similar to the proof of Lemma 14.

$\square$

r	lower bounds on size	Reasons
3	$q_{3,\Delta,t}+6t+6-\Delta$ or $q_{3,\Delta,t}+7t+2$	Corollary 12(ii), Lemma 13
2	$q_{3,\Delta,t}+6t+6-\Delta$ or $8t+5$	Corollary 12(ii), Lemma 13
1	$q_{4,\Delta,t}+4t+3-\Delta$	Corollary 12(i), Lemma 13

Table 2: Lower bounds on size of G in the proof of Lemma 15

Corollary 16. *(1) $q_{4,6,t}$ is at least $5t-3$ and $4t+3$ for t at least 3 and 6, respectively.*
(2) $q_{4,7,t}$ is at least $5t-4$ and $4t+2$ for t at least 3 and 6, respectively.
(3) $q_{5,6,t}$ is at least $9t-6$ and $8t$ for t at least 3 and 6, respectively.
(4) $q_{5,7,t}$ is at least $9t-8$ and $8t-2$ for t at least 3 and 6, respectively.

Proof. The results can be calculated directly from Lemmas 13 to 15. □

Corollary 17. *Each C_3-free planar graph G with maximum degree at most 7 and $|G| \geq 18$ has an equitable 6-coloring. Moreover, each C_3-free planar graph G with maximum degree 6 has an equitable 6-coloring.*

Proof. Let G be an edge-minimal C_3-free planar graph that is not equitably Δ-colorable with $|G| = 6t$, where t is an integer at least 3, and maximum degree at most 7.

Consider the case $r = 3$. By Corollaries 12(ii) and 16, $e(G) > \min\{2q_{3,\Delta,t}+9t+3-\Delta, q_{3,\Delta,t}+11t+2\} \geq 13t-4 \geq 12t-4$.

Consider the case $r = 2$. By Corollary 12(i), $e(G) > q_{4,\Delta,t}+8t+6-\Delta$. It follows from Corollary 16 that $e(G) > \min\{13t-5, 12t+1\} \geq 12t-4$ for $t \geq 3$.

Consider the case $r = 1$. We have $e(B', V_1) \geq 5t$ by Observation 2. But y has at most $\Delta - 1$ neighbors in B' because $xy \in E(G)$, so $(t-1)\Delta - 1 \geq e(B', V_1)$. Consequently, $(t-1)\Delta - 1 \geq 5t$. That is $t \geq 4$ when $\Delta \leq 7$. By Corollary 12 (i), $e(G) > q_{5,\Delta,t}+5t-4$. Using Corollary 16, we have $e(G) > \min\{14t-12, 13t-6\}$. It follows from $t \geq 4$ that $e(G) > 12t-4$.

Since we have contradiction for all cases, the counterexample is impossible. Use Lemma 4 to complete the first part of the proof.

Observation 6 implies each C_3-free planar graph G with maximum degree 6 has an equitable 6-coloring. $\square$

Note that a graph G in Corollary 17 has an equitable m-coloring with $m < \Delta(G)$.

Lemma 18. *Each C_3-free planar graph G with maximum degree at most 7 has an equitable 7-coloring.*

Proof. Use Table 3 for an argument similar to the proof of Lemma 14.

r	lower bounds on size	Reasons
3	$q_{3,\Delta,t} + 12t + q_{4,\Delta,t} + 3 - \Delta$	Corollary 12(i), Lemma 13
2	$q_{5,\Delta,t} + 10t + 6 - \Delta$	Corollary 12(i), Lemma 13
1	$q_{6,\Delta,t} + 6t + 3 - \Delta$	Corollary 12(i), Lemma 13

Table 3: Lower bounds on size of G in the proof of Lemma 18

$\square$

Using Corollary 16, and $q_{6,\Delta,t} = 12t-4$ from Corollary 17, we have $e(G) > 14t-4$ for each case of r, which is a contradiction. Thus the counterexample is impossible. Use Observation 6 to complete the proof. $\square$

Theorem 19. *Each C_3-free planar graph G with maximum degree $\Delta \geq 6$ has an equitable Δ-coloring.*

Proof. Zhu and Bu [15] proved that the theorem holds for every C_3- free planar graph with maximum degree at least 8. Use Corollary 17 and Lemma 18 to complete the proof. $\square$

Next, we show that the conjecture holds also for a planar graph of maximum degree 5 if we restrict the girth to be at least 6.

Corollary 20. *Each planar graph G of girth at least 6 with maximum degree at most 6 and $|G| \geq 15$ has an equitable 5-coloring. Moreover, each planar graph G with girth at least 6 and maximum degree $\Delta \geq 5$ has an equitable Δ-coloring.*

Proof. Let G be an edge-minimal planar graph of girth at least 6 that is not equitably Δ-colorable with $|G| = 5t$, where t is an integer at least 3, and maximum degree at most 6.

Then for $t \geq 3$, we have $e(G) \leq (15/2)t - 3$ from Lemma 1, and $e(G) > \min\{9t - 6, 8t\}$ from Corollary 16 which leads to a contradiction. Thus the counterexample is impossilble. Use Lemma 4 to complete the first part of the proof.

Observation 6 implies each planar graph G with girth at least 6 and maximum degree 5 has an equitable 5-coloring. Use Theorem 19 to complete the proof. $\square$

4 Results on Planar Graphs without C_4

First we introduce the result by Tan [11].

Lemma 21. *If a planar graph G of order n does not contain C_4, then $e(G) \leq (15/7)n - (30/7)$ and $\delta(G) \leq 4$.*

The proof of Lemma 21 by Tan is presented here for convenience of readers.

Proof. Let f and f_i denote the number of faces and the number of faces of length i, respectively. We need only to consider the case that G is connected. A graph G cannot contain two C_3 that share the same edge since G does not contain C_4. It follows that $3f_3 \leq e(G)$.

Consider $5f - 2f_3 = 5(f_3 + f_5 + \cdots + f_n) - 2f_3 \leq 3f_3 + 5f_5 + \cdots + nf_n = \sum_{1 \leq i \leq n} if_i = 2e(G)$. Thus $f \leq (8/15)e(G)$. Using Euler's formula, we have $e(G) \leq (15/7)n - (30/7)$. The result about minimum degree follows from Handshaking Lemma. $\square$

From Lemma 21, each edge-minimal counterexample graph has $1 \leq r \leq 4$. The following tools in this section are quite similar to that of the previous section. Thus we omit the proofs of them.

Lemma 22. *Let m be a fixed integer with $m \geq 1$. Suppose that any planar graph without C_4 of order mt with maximum degree at most Δ is equitably m-colorable*

for any integer $t \geq k$. Then any planar graph without C_4 of order at least kt and maximum degree at most Δ is also equitably m-colorable.

Observation 23. *By Lemmas 5 and 22, for proving that the conjecture holds for planar graphs without C_4 it suffices to prove only planar graphs without C_4 of order Δt where $t \geq 3$ is a positive integer.*

Lemma 24. *Suppose G is a planar graph without C_4 with $\Delta(G) = \Delta$. If G has an independent s-set V' and there exists $U \subseteq V(G) - V'$ such that $|U| > s(2+\Delta)/2$ and $e(u, V') \geq 1$ for all $u \in U$, then U contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$.*

Notation. *Let $p_{m,\Delta,t}$ denote the maximum number not exceeding $(15/7)mt - (30/7)$ such that each planar graph without C_4 of order mt, where t is an integer, is equitably m-colorable if it has maximum degree at most Δ and size at most $p_{m,\Delta,t}$.*

Lemma 25. *Let G be an edge-minimal planar graph without C_4 that is not equitably m-colorable with order mt, where t is an integer, and maximum degree at most Δ. If $e(G) \leq (r+1)(m-r)t - t + 2 + p_{r,\Delta,t}$, then B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1$.*

Lemma 26. *Let G be an edge-minimal planar graph without C_4 that is not equitably m-colorable with order mt, where t is an integer, and maximum degree at most Δ. If B contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V_1$, then $e(G) \geq r(m-r)t + p_{r,\Delta,t} + p_{m-r,\Delta,t} - \Delta + 4$.*

Corollary 27. *Let G be an edge-minimal planar graph without C_4 that is not equitably m-colorable with order mt, where t is an integer, and maximum degree at most Δ. Then $e(G) \geq r(m-r)t + p_{r,\Delta,t} + p_{m-r,\Delta,t} - \Delta + 4$ if one of the following conditions are satisfied:*

(i) $(m-r)t + 1 > (t-1)(2+\Delta)/2$;

(ii) $e(G) \leq (r+1)(m-r)t - t + 2 + p_{r,\Delta,t}$.

Now we are ready to work on planar graphs without C_4.

Lemma 28. *(i)* $p_{1,\Delta,t} = 0$. *(ii)* $p_{2,\Delta,t} = 2$. *(iii)* $p_{3,\Delta,t} \geq 6$ *for* $t \geq 3$. *(iv)* $p_{4,\Delta,t} \geq 3t$.

Proof. (i), (ii) and (iii) are obvious. (iv) is the result of Lemma 7. □

Lemma 29. $p_{5,\Delta,t} \geq \min\{p_{4,\Delta,t} + 16t + 3 - \Delta, 6t + 11 - \Delta, 7t + 2\}$ *for* $\Delta \geq 8$ *and* $t \geq 3$.

Proof. Use Table 4 for an argument similar to the proof of Lemma 14.

r	lower bounds on size	Reasons
4	$p_{4,\Delta,t} + 16t + 3 - \Delta$ or $p_{4,\Delta,t} + 4t + 2$	Corollary 27(ii), Lemma 28
3	$6t + 11 - \Delta$ or $7t + 8$	Corollary 27(ii), Lemma 28
2	$6t + 11 - \Delta$ or $8t + 4$	Corollary 27(ii), Lemma 28
1	$p_{4,\Delta,t} + 4t + 3 - \Delta$ or $7t + 2$	Corollary 27(ii), Lemma 28

Table 4: Lower bounds on size of G in the proof of Lemma 29

□

Lemma 30. $p_{6,\Delta,t} \geq \min\{p_{4,\Delta,t} + 8t + 5 - \Delta, 9t + 15 - \Delta, 11t + 4, p_{5,\Delta,t} + 5t + 3 - \Delta\}$ *for* $\Delta \geq 8$ *and* $t \geq 3$.

Proof. Use Table 5 for an argument similar to the proof of Lemma 14.

r	lower bounds on size	Reasons
4	$p_{4,\Delta,t} + 8t + 5 - \Delta$ or $p_{4,\Delta,t} + 9t + 2$	Corollary 27(ii), Lemma 28
3	$9t + 15 - \Delta$ or $11t + 8$	Corollary 27(ii), Lemma 28
2	$p_{4,\Delta,t} + 8t + 5 - \Delta$ or $11t + 4$	Corollary 27(ii), Lemma 28
1	$p_{5,\Delta,t} + 5t + 3 - \Delta$	Corollary 27(i), Lemma 28

Table 5: Lower bounds on size of G in the proof of Lemma 30

□

Corollary 31. *(1) $p_{5,8,t}$ is at least $7t-5$ and $6t+3$ for t at least 3 and 8, respectively.*
(2) $p_{6,8,t}$ is at least $12t-10$ and $9t+7$ for t at least 3 and 6, respectively.
(3) $p_{7,8,t}$ is at least $18t-15$ and $15t+1$ for t at least 3 and 6, respectively.

Proof. The results can be calculated directly from Lemmas 28 to 30. □

Corollary 32. *Each planar graph G without C_4 with maximum degree at most 8 and $|G| \geq 21$ has an equitable 7-coloring. Moreover, each planar graph G without C_4 with maximum degree 7 has an equitable 7-coloring.*

Proof. Let G be an edge-minimal planar graph without C_4 that is not equitably Δ-colorable with $|G| = 7t$, where t is an integer at least 3, and maximum degree at most 8.

Consider the case $r = 4$. By Corollaries 27(ii) and 31, $e(G) > \min\{p_{4,\Delta,t} + 12t + p_{3,\Delta,t} + 3 - \Delta, p_{4,\Delta,t} + 14t + 2\} \geq 15t + 1 \geq 15t - (30/7)$ for $t \geq 3$.

Consider the case $r = 3$. By Corollaries 27(ii) and 31, $e(G) > \min\{p_{3,\Delta,t} + 12t + p_{4,\Delta,t} + 3 - \Delta, p_{3,\Delta,t} + 15t + 2\} \geq 15t + 1 \geq 15t - (30/7)$ for $t \geq 3$.

Consider the case $r = 2$. By Corollaries 27(i) and 31, $e(G) > 10t + p_{5,\Delta,t} + 3 - \Delta \geq 15t - (30/7)$ for $t \geq 3$.

Consider the case $r = 1$. We have $e(B', V_1) \geq 6t$ by Observation 2. But y has at most $\Delta - 1$ neighbors in B' because $xy \in E(G)$, so $(t-1)\Delta - 1 \geq e(B', V_1)$. Consequently, $(t-1)\Delta - 1 \geq 5t$. That is $t \geq 4.5$ when $\Delta \leq 8$. By Corollary 27 (i), $e(G) > p_{6,\Delta,t} + 6t - 5$. Using Corollary 31, we have $e(G) > \min\{18t - 15, 15t + 1\}$. It follows from $t \geq 4.5$ that $e(G) > 15t - (30/7)$.

Since we have contradiction for all cases, the counterexample is impossible. Use Lemma 22 to complete the first part of the proof.

Observation 23 implies each planar graph G without C_4 with maximum degree 7 has an equitable 7-coloring. □

Lemma 33. *Each planar graph G without C_4 with maximum degree at most 8 has an equitable 8-coloring.*

r	lower bounds on size	Reasons
4	$p_{4,\Delta,t} + 16t + p_{4,\Delta,t} + 3 - \Delta$ or $p_{4,\Delta,t} + 19t + 2$	Corollary 27(ii), Lemma 28
3	$15t + p_{5,\Delta,t} + 9 - \Delta$	Corollary 27(i), Lemma 28
2	$p_{6,\Delta,t} + 12t + 5 - \Delta$	Corollary 27(i), Lemma 28
1	$p_{7,\Delta,t} + 7t + 3 - \Delta$	Corollary 27(i), Lemma 28

Table 6: Lower bounds on size of G in the proof of Lemma 33

Proof. Use Table 6 for an argument similar to the proof of Lemma 14.

□

Using Corollary 31, and $q_{7,\Delta,t} = 15t - 4$ from Corollary 32, we have $e(G) > (120/7)t - (30/7)$ for each case of r, which is a contradiction. Thus the counterexample is impossible. Use Observation 23 to complete the proof. □

Theorem 34. *Each planar graph G without C_4 with maximum degree $\Delta \geq 7$ has an equitable Δ-coloring.*

Proof. Nakprasit [10] proved that the theorem holds for every planar graph with maximum degree at least 9. Use Corollary 32 and Lemma 33 to complete the proof.

□

5 Acknowledgement

The first author was supported by Research Promotion Fund, Khon Kaen University, Fiscal year 2011.

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